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AN ASYMPTOTIC BEHAVIOR OF THE MEAN OF SOME EXPONENTIAL FUNCTIONALS OF A RANDOM WALK

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1. Introduction

Let $\{\xi_i\}_{i\geq 1}$ be a sequence of independent identically distributed random variables, and set $S_0 = 0$, $S_n = \sum_{i=1}^{n} \xi_i$, $n \geq 1$. Let $f(x)$, $h(x)$ be two functions on $[0,\infty)$. For $n \geq 1$, set

$$P_n = E\left[f(e^{S_n})h\left(\sum_{i=1}^{n} e^{S_i}\right)\right].$$

One object of this paper is to obtain an estimate of P_n under certain conditions of ξ_1, $f(x)$ and $h(x)$.

We can consider a similar problem in case of Brownian motion. Let $\{B(t) : t \geq 0\}$ be a Brownian motion with $B(0) = 0$. We assume that $f(x)$ and $h(x)$ satisfy the following: for some $0 < \alpha < \beta < \infty$,

$$\sup_{x\geq 0} x^{-\alpha}|f(x)| < \infty, \qquad \sup_{x\geq 0} x^{\beta}|h(x)| < \infty.$$

Set

$$P(t) = E\left[f(e^{B(t)})h\left(\int_0^t e^{B(s)}ds\right)\right].$$

Applying a very useful formula of Yor ([9],(6.e)), we see as in [5] that

$$P(t) \sim ct^{-3/2} \quad \text{as } t \to \infty,$$

where

$$c = 2^{\frac{5}{2}}\pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_0^\infty f(y^2)h(4/z)e^{-\lambda z}u \sinh u\,dydzdu, \quad \lambda = (1+y^2)/2 + y\cosh u.$$

Kawazu-Tanaka [5] used this fact to obtain the rate of decay of the tail probability of the maximum of a diffusion process in a drifted Brownian environment. This result in case of a Brownian motion suggests that an analogous result in a random walk case also holds. In fact we obtain Theorem 1 below. But to our regret, we can not find the precise value corresponding to c.

The original motivation of this problem is to clarify and simplify the proof of Afanas'ev's result [1] on a random walk in a random medium. In the last section we get Theorem 2 which is a slight extention of the theorem in [1] by using our Theorem 1.

Now we state the conditions on $\xi = \xi_1$ and Theorem 1.

Conditions (a)-(d).

(a) $E\xi = 0$, $0 < E\xi^2 = \sigma^2 < \infty$.

(b) The distribution of ξ is continuous and $E|\xi|^3 < \infty$.

(c) $Ee^{\theta\xi}$ converges for some $\theta > 0$.

The condition (b) can be replaced by the following condition (d).

(d) The distribution of ξ is concentrated on the set $\{id : i \in \mathbb{Z}\}$ with some $d > 0$.

Theorem 1. *Let functions f, h and W satisfy the following conditions:*

(i) *f and h are continuous on $[0, \infty)$.*

(ii) *W is continuous on $\mathbb{R}$ and non-negative.*

(iii) *There exist positive numbers α, β, ε, η such that*

$$\sup_{x \geq 0} x^{-\alpha}|f(x)| < \infty, \quad \sup_{x \geq 0} x^{\beta}|h(x)| < \infty,$$
$$\limsup_{x \to -\infty} W(x)e^{-\varepsilon x} < \infty, \quad \liminf_{x \to +\infty} W(x)e^{-\eta x} > 0,$$

with $\beta\eta > \alpha > 0$. If the conditions (a), (b) *and* (c) *are satisfied, then*

$$E\left[f(e^{S_n})h\left(\sum_{i=1}^{n} W(S_i)\right)\right] \sim cn^{-3/2},$$

as $n \to \infty$ with a constant c. The same conclusion holds under the conditions (a), (d) *and* (c).

REMARK. A sufficient condition for the positivity of c is given in Corollary 10.

2. Preliminaries

We introduce the following quantities together with their corresponding Laplace transforms.

$$u_n(x) = P(S_1 > 0, \cdots, S_{n-1} > 0, 0 < S_n \leq x),$$

$$
\begin{aligned}
v_n(x) &= P(S_1 \le 0, \cdots, S_{n-1} \le 0, -x \le S_n \le 0), \\
u_0(x) &= v_0(x) = 1_{[0,\infty)}(x), \\
\varphi_n(\theta) &= \int_0^\infty e^{-\theta x} du_n(x), \quad \psi_n(\theta) = \int_0^\infty e^{-\theta x} dv_n(x).
\end{aligned}
$$

The three lemmas below will be used later. We state them without proof.

Lemma 1 (Spitzer-Baxter identity. See [6], p. 49). *For any $\theta > 0$, $|t| < 1$,*

$$
1 + \sum_{n=1}^\infty \varphi_n(\theta) t^n = \exp\left\{ \sum_{n=1}^\infty \frac{t^n}{n} E(e^{-\theta S_n}; S_n > 0) \right\}.
$$

Lemma 2 (See [2]). *If the conditions* (a) *and* (b) *hold, then for any $\theta > 0$,*

$$
E(e^{-\theta S_n}; S_n > 0) \sim (\sqrt{2\pi}\sigma\theta)^{-1} n^{-\frac{1}{2}} \quad \text{as } n \to \infty.
$$

Lemma 3 (See [4] and [8]).

(1) *Let $\sum_{n=0}^\infty a_n t^n = \exp(\sum_{n=1}^\infty b_n t^n)$ for $|t| < 1$. If $b_n \sim bn^{-3/2}$, then $a_n \sim (b \exp B) n^{-3/2}$ with $B = \sum_{n=1}^\infty b_n$.*

(2) *Let $c_n \ge 0$, $d_n \ge 0$, $c_n \sim cn^{-3/2}$ and $d_n \sim dn^{-3/2}$. If $a_n = \sum_{j=0}^n c_{n-j} d_j$, then $a_n \sim (cD + dC) n^{-3/2}$ with $C = \sum_{n=0}^\infty c_n$ and $D = \sum_{n=0}^\infty d_n$.*

Set

$$
u(x) = \sum_{n=0}^\infty u_n(x), \qquad v(x) = \sum_{n=0}^\infty v_n(x),
$$

and

$$
\begin{aligned}
U(x) &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^x u(y) dy, \qquad \varphi(\theta) = \int_0^\infty e^{-\theta x} dU(x), \\
V(x) &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^x v(y) dy, \qquad \psi(\theta) = \int_0^\infty e^{-\theta x} dV(x).
\end{aligned}
$$

Then we can prove the following lemma.

Lemma 4. *If the conditions* (a) *and* (b) *hold, then*

(1) $\lim_{n\to\infty} n^{3/2} \varphi_n(\theta) = \varphi(\theta)$, $\lim_{n\to\infty} n^{3/2} \psi_n(\theta) = \psi(\theta)$ *for each $\theta > 0$.*

(2) $\lim_{n\to\infty} n^{3/2} u_n(x) = U(x)$, $\lim_{n\to\infty} n^{3/2} v_n(x) = V(x)$ *compact uniformly on $[0, \infty)$.*

(3) $\lim_{n\to\infty} n^{3/2} P(M_n \le x, M_n - S_n \le y) = U(x) v(y) + u(x) V(y)$ *where $M_n = \max_{0 \le j \le n} S_j$.*

Proof. Combining Lemmas 1, 2 with (1) of Lemma 3, for each $\theta > 0$, we have

$$\lim_{n\to\infty} n^{\frac{3}{2}}\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}\sigma\theta}\left(1+\sum_{n=1}^{\infty}\varphi_n(\theta)\right).$$

It is easy to see that the right hand side of the above is $\varphi(\theta)$. If we consider a reversed random walk $\{-S_n\}_{n\geq 0}$, we get the assertion for ψ also. By the extended continuity theorem for Laplace transform, (1) yields (2). In view of (2), (3) can be proved by applying (2) of Lemma 3 to the following identity ([6], p. 25).

$$P(M_n \leq x, M_n - S_n \leq y) = \sum_{j=0}^{n} u_j(x)v_{n-j}(y) \qquad n \geq 0. \tag{2.1}$$

The proof of Lemma 4 is complete. □

We give some definitions. Fix a constant $K > 0$. For a function f on $(-\infty, K]$ and a constant $a \in \mathbb{R}$, set

$$\|f\|_a = \sup_{x\leq K} e^{ax}|f(x)|,$$

and define a function space $C(a)$:

$$C(a) = \{f : \ f \text{ is continuous on } (-\infty, K] \text{ and } \|f\|_a < \infty\}.$$

$C(a)$ is a Banach space with respect to the norm $\| \ \|_a$. From now on we omit a in $\| \ \|_a$. By the condition (c), we can take $\delta > 0$ such that $E(e^{\delta\xi}) < \infty$. Let this $\delta > 0$ and arbitrary $\mu > \delta$ be fixed, and set $\lambda = \mu - \delta > 0$. Let non-negative $W \in C(-\mu)$ be fixed. For any $f \in C(\lambda)$, we define a family of transforms $\{R_t\}_{t\geq 0}$ depending on W by

$$R_t f(x) = E_x\left[f(\xi)\{1 - e^{-tW(\xi)}\}; \xi \leq K\right], \tag{2.2}$$

where E_x denotes the expectation of the random walk starting at x. For any $f \in C(-\delta)$, we define a sequence of transforms $\{T_n\}_{n\geq 0}$ by

$$T_n f(x) = E_x\left[f(S_n); M_n \leq K\right], \qquad n \geq 0. \tag{2.3}$$

In the sequel, the conditions (a)-(c) are assumed to be satisfied.

Lemma 5. *Each R_t is a bounded operator from $C(\lambda)$ to $C(-\delta)$.*

Proof. Let $f \in C(\lambda)$. The continuity of $R_t f$ is derived from the condition (b) and the continuity of f and W. From the assumptions of f and W,

$$|f(x)\{1 - e^{-tW(x)}\}| \leq tW(x)|f(x)| \leq t\|W\|\|f\|e^{(\mu-\lambda)x}.$$

By the definition of λ and (2.2),

$$\begin{aligned}|R_t f(x)| &\leq t\|W\|\|f\|E(e^{\delta(x+\xi)}; \xi \leq K-x)\\ &\leq t\|W\|\|f\|e^{\delta x}E(e^{\delta\xi}).\end{aligned}$$

Therefore we obtain

$$\|R_t\| \leq t\|W\|E(e^{\delta\xi}) < \infty.$$

The proof of Lemma 5 is complete. □

Lemma 6. *Each T_n is a bounded operator from $C(-\delta)$ to $C(\lambda)$, and there exists a constant C such that $\|T_n\| \leq Cn^{-3/2}$ for $n \geq 1$. Here C does not depend on n.*

Proof. Let $f \in C(-\delta)$. The continuity of $T_n f$ is derived from the condition (b) and the continuity of f. Direct calculations show $\|T_0\| = e^{\mu K}$. We will estimate $\|T_n\|$ for $n \geq 1$. Since $f \in C(-\delta)$, we have $|f(x)| \leq \|f\|e^{\delta x}$. Applying this and (2.1) to (2.3), we see

$$\begin{aligned}|T_n f(x)| &\leq \|f\|e^{\delta x}E(e^{\delta S_n}; M_n \leq K-x)\\ &\leq \|f\|e^{\delta x}\sum_{j=0}^{n}\psi_{n-j}(\delta)\int_0^{K-x} e^{\delta y}du_j(y).\end{aligned}$$

Using Chebyshev's inequality, we see

$$\int_0^{K-x} e^{\delta y}du_j(y) \leq e^{\delta(K-x)}u_j(K-x) \leq e^{(\delta+\lambda)(K-x)}\varphi_j(\lambda).$$

Thus we have

$$|T_n f(x)| \leq \|f\|e^{\mu K-\lambda x}\sum_{j=0}^{n}\psi_{n-j}(\delta)\varphi_j(\lambda),$$

which shows

$$\|T_n\| \leq e^{\mu K}\sum_{j=0}^{n}\psi_{n-j}(\delta)\varphi_j(\lambda).$$

Applying (2) of Lemma 3 and (1) of Lemma 4 to the right hand side of the above, we have

$$C := e^{\mu K}\sup_{n\geq 1}\left\{n^{\frac{3}{2}}\sum_{j=0}^{n}\psi_{n-j}(\delta)\varphi_j(\lambda)\right\} < \infty.$$

This completes the proof of Lemma 6. □

From Lemma 6 we get the following corollary.

Corollary 7. *Let $S = \sum_{n=0}^{\infty} T_n$. Then S is a bounded operator from $C(-\delta)$ to $C(\lambda)$.*

Now we consider the limit of $n^{3/2}T_n$. Let $f \in C(-\delta)$, $x \leq K$ and $L > 0$ be fixed. Then

$$\begin{aligned} T_n f(x) &= E_x\left[f(S_n); M_n \leq K, M_n - S_n \leq L\right] \\ &\quad + E_x\left[f(S_n); M_n \leq K, M_n - S_n > L\right] = a_n + b_n. \end{aligned}$$

Applying (3) of Lemma 4 to a_n, we have

$$\lim_{n\to\infty} n^{\frac{3}{2}} a_n = \int_0^L \int_0^{K-x} f(x+a-b)d(U(a)v(b) + u(a)V(b)).$$

Using the method employed in the proof of Lemma 6, we get

$$|b_n| \leq \|f\| e^{\mu K - \lambda x} \sum_{j=0}^{n} \varphi_{n-j}(\lambda) \int_L^\infty e^{-\delta b} dv_j(b).$$

In view of (1) and (2) of Lemma 4, we apply (2) of Lemma 3 to the right hand side

$$\limsup_{n\to\infty} n^{\frac{3}{2}} |b_n| \leq \|f\| e^{\mu K - \lambda x} \left\{ \sum_{n=0}^{\infty} \varphi_n(\lambda) \int_L^\infty e^{-\delta b} dV(b) + \varphi(\lambda) \int_L^\infty e^{-\delta b} dv(b) \right\}.$$

The last term goes to 0 as $L \to \infty$. Without loss of generality, we assume $f \geq 0$. Then we have

$$n^{\frac{3}{2}} a_n \leq n^{\frac{3}{2}} T_n f(x) \leq n^{\frac{3}{2}} a_n + \sup_{m\geq n} (m^{\frac{3}{2}} b_m).$$

Going to the limit, first with respect to n, and then with respect to L, we obtain

$$\lim_{n\to\infty} n^{\frac{3}{2}} T_n f(x) = \int_0^\infty \int_0^{K-x} f(x+a-b)d(U(a)v(b) + u(a)V(b)). \tag{2.4}$$

Denote by $Qf(x)$ the right hand side of (2.4). It is clear that Qf is continuous by (2) and (3) of Lemma 4. The definition of Q and Lemma 6 show that

$$|Qf(x)| \leq \limsup_{n\to\infty} n^{\frac{3}{2}} |T_n f(x)| \leq \left(\limsup_{n\to\infty} n^{\frac{3}{2}} \|T_n\|\right) \|f\| e^{-\lambda x}.$$

Hence we have

$$\|Q\| \leq \limsup_{n\to\infty} \left(n^{\frac{3}{2}} \|T_n\|\right) \leq C.$$

Collecting the above results, we get the following lemma.

Lemma 8. *Let $f \in C(-\delta)$. Then we can define $Qf(x) = \lim_{n\to\infty} n^{3/2} T_n f(x)$ and Q is a bounded operator from $C(-\delta)$ to $C(\lambda)$.*

The following corollary will be used hereafter.

Corollary 9. *If $f \in C(-\delta)$ is positive, then $Qf(0) > 0$.*

Proof. From (2.4) and the assumption of f, we have

$$Qf(0) \geq \int_0^\infty \int_0^K f(a-b)dU(a)dv(b).$$

By the definition of $U(x)$, we see $U(K) \geq K/\sqrt{2\pi}\sigma > 0$. The renewal theory ([3], Chap. XI) applied to $v(x)$ yields $\lim_{x\to\infty} v(x) = \infty$, which shows the corollary. □

3. Main Lemma

The lemmas established in the previous section enable us to show Theorem 1 if $h(x) = e^{-tx}$ ($t \geq 0$) and $W(x) = \infty$ for $x > K$. That is, we have the following lemma.

Main Lemma. *For any $f \in C(-\delta)$, non-negative $W \in C(-\mu)$ and $t \geq 0$, the finite limit*

$$\lim_{n\to\infty} n^{3/2} E\left[f(S_n)\exp\left\{-t\sum_{i=1}^n W(S_i)\right\}; M_n \leq K\right]$$

exists.

To prove Main Lemma we need some preparations. Let non-negative $W \in C(-\mu)$ be fixed. For each $f \in C(-\delta)$ and $t \geq 0$, we put

$$A_0^{(t)}f(x) = f(x),\ \ A_n^{(t)}f(x) = E_x\left[f(S_n)\exp\left\{-t\sum_{i=1}^n W(S_i)\right\}; M_n \leq K\right],\ \ n \geq 1.$$

Then $A_n^{(t)} = \{A_1^{(t)}\}^n$ as a transform, for $\{\xi_i\}_{i\geq 1}$ is i.i.d. (2.2) and (2.3) yield $T_1 - A_1^{(t)} = R_t$. Put $f_n^{(t)} = A_n^{(t)} f$. Then $f_n^{(t)} = A_1^{(t)} f_{n-1}^{(t)} = (T_1 - R_t) f_{n-1}^{(t)}$. By induction we get the following equations in $C(\lambda)$.

$$f_0^{(t)} = f, \quad f_n^{(t)} = T_n f - \sum_{j=1}^{n} T_{n-j} R_t f_{j-1}^{(t)} \quad n \geq 1. \tag{3.1}$$

Taking account of the definitions of $A_n^{(t)}$, T_n and Lemma 6, we have

$$\|f_n^{(t)}\| \leq \|T_n|f|\| \leq C\|f\| n^{-\frac{3}{2}}. \tag{3.2}$$

The operator SR_t maps $C(\lambda)$ to $C(\lambda)$. Hence we can define

$$c_1 = \sup\{t > 0 : \|SR_t\| < 1\}.$$

For a while we assume $0 \leq t < c_1$ and omit t in R_t and $f_n^{(t)}$. Then there exists $(1+SR)^{-1}$. By (3.1) and (3.2), $F \equiv \sum_{n=0}^{\infty} f_n = (1+SR)^{-1} Sf$. Let $g \in C(\lambda)$ satisfy the equation

$$g = Qf - QRF - SRg. \tag{3.3}$$

After some tedius calculations, we rewrite g as follows.

$$g = (1+SR)^{-1} Q (1+RS)^{-1} f.$$

We set

$$\bar{g}(x) = \limsup_{n\to\infty} |n^{\frac{3}{2}} f_n(x) - g(x)|.$$

By (3.2), $\|\bar{g}\| \leq C\|f\| + \|g\| < \infty$. Now we can start the proof of Main Lemma.

Proof. For fixed $N \in \mathbb{N}$, we have

$$n^{\frac{3}{2}} \sum_{j=1}^{n} T_{n-j} R f_{j-1} = n^{\frac{3}{2}} \left[\sum_{j=1}^{N} + \sum_{j=N+1}^{n-N} + \sum_{j=n-N+1}^{n} \right] T_{n-j} R f_{j-1} = I_n + J_n + K_n.$$

Combining (3.1) and (3.3) with the above, we see

$$|n^{\frac{3}{2}} f_n - g| \leq |n^{\frac{3}{2}} T_n f - Qf| + |I_n - QRF| + |K_n - SRg| + |J_n|. \tag{3.4}$$

We estimate each term of the right hand side of (3.4). Using Lemmas 5, 6 and 8, we have

$$I_n = \sum_{j=1}^{N} \left(\frac{n}{n-j} \right)^{\frac{3}{2}} (n-j)^{\frac{3}{2}} T_{n-j} R f_{j-1} \longrightarrow \sum_{j=1}^{N} QRf_{j-1} \quad \text{as } n \to \infty.$$

Therefore we get

$$\lim_{n\to\infty}(QRF - I_n) = \sum_{j=N}^{\infty} QRf_j. \tag{3.5}$$

Rewriting K_n, we have

$$K_n = n^{\frac{3}{2}} \sum_{j=1}^{N} T_{j-1}Rf_{n-j} = \sum_{j=1}^{N} \left(\frac{n}{n-j}\right)^{\frac{3}{2}} T_{j-1}R\left\{(n-j)^{\frac{3}{2}} f_{n-j}\right\}.$$

By Corollary 7,

$$|K_n - SRg| \le \left| K_n - \sum_{j=1}^{N} T_{j-1}Rg \right| + \sum_{j=N+1}^{\infty} T_{j-1}R|g|.$$

The first term of the right hand side is bounded from above by the quantity

$$\sum_{j=1}^{N} \left(\frac{n}{n-j}\right)^{\frac{3}{2}} T_{j-1}R\left|(n-j)^{\frac{3}{2}} f_{n-j} - g\right| + \sum_{j=1}^{N} \left\{ \left(\frac{n}{n-j}\right)^{\frac{3}{2}} - 1 \right\} T_{j-1}R|g|.$$

Using Fatou's lemma, we see

$$\limsup_{n\to\infty} |K_n - SRg| \le \sum_{j=0}^{N-1} T_jR\bar{g} + \sum_{j=N}^{\infty} T_jR|g|. \tag{3.6}$$

Here we note that $T_jR\bar{g}$ is well-defined. Applying Lemmas 5, 6 and (3.2) to J_n,

$$\|J_n\| \le C^2 \|R\| \|f\| n^{\frac{3}{2}} \sum_{j=N+1}^{n-N} (n-j)^{-\frac{3}{2}} (j-1)^{-\frac{3}{2}}.$$

Using (2) of Lemma 3, we have

$$\limsup_{n\to\infty} \|J_n\| \le 2C^2 \|R\| \|f\| \sum_{j=N}^{\infty} j^{-\frac{3}{2}}. \tag{3.7}$$

Collecting (3.4)-(3.7) and letting $N \to \infty$, we have $0 \le \bar{g} \le SR\bar{g}$. Thus $\|\bar{g}\| \le \|SR_t\| \|\bar{g}\|$. Since $t \in [0, c_1)$, this estimate implies $\|\bar{g}\| = 0$, i.e., $\bar{g} \equiv 0$, which shows the following: If $0 \le t < c_1$, then for each $f \in C(-\delta)$ and $x \le K$,

$$\lim_{n\to\infty} n^{\frac{3}{2}} A_n^{(t)} f(x) = (1 + SR_t)^{-1} Q (1 + R_tS)^{-1} f(x). \tag{3.8}$$

If $c_1 = \infty$, this completes the proof of Main Lemma. We consider the case of $c_1 < \infty$. Let arbitrary $t_0 \in (0, c_1)$ be fixed. For $f \in C(-\delta)$ and $x \leq K$, set

$$\overline{T}_n f(x) = A_n^{(t_0)} f(x) \quad n \geq 0 \text{ and } \quad \overline{S} = \sum_{n=0}^{\infty} \overline{T}_n.$$

Since $0 \leq \overline{T}_n f(x) \leq T_n f(x)$ if $f \geq 0$, everything of the above is well-defined. For $f \in C(\lambda)$ and $t \geq 0$, set

$$\overline{R}_t f(x) = E_x \left[f(\xi) e^{-t_0 W(\xi)} \{1 - e^{-tW(\xi)}\}; \xi \leq K \right].$$

Let $f \in C(-\delta)$. In view of (3.8), we define $\overline{Q} f(x)$ as follows.

$$\overline{Q} f(x) = \lim_{n \to \infty} n^{\frac{3}{2}} \overline{T}_n f(x) = (1 + SR_{t_0})^{-1} Q (1 + R_{t_0} S)^{-1} f(x).$$

Now exactly the same argument as the above is possible if we replace T_n, R_t, Q by $\overline{T}_n$, $\overline{R}_t$, $\overline{Q}$ respectively. Therefore setting $c_2 = \sup\{t > 0 : \|\overline{SR}_t\| < 1\}$ where $\overline{SR}_t = \overline{S}(\overline{R}_t)$, we get the following: If $0 < t \leq c_2$, then

$$\lim_{n \to \infty} n^{\frac{3}{2}} A_n^{(t+t_0)} f(x) = (1 + \overline{SR}_t)^{-1} \overline{Q} (1 + \overline{R_t S})^{-1} f(x). \tag{3.9}$$

On the other hand, we have the following relations from the definitions.

$$\overline{R}_t = R_{t+t_0} - R_{t_0}, \quad \overline{S} = S - SR_{t_0} \overline{S}. \quad (\text{cf. } (3.1))$$

Using the above relations, we get

$$\begin{aligned} (1 + SR_{t_0})(1 + \overline{SR}_t) &= (1 + SR_{t_0+t}), \\ (1 + \overline{R_t S})(1 + R_{t_0} S) &= (1 + R_{t_0+t} S). \end{aligned}$$

From these identities and the definition of $\overline{Q}$, we see that the right hand side of (3.9) is equivalent to $(1 + SR_{t_0+t})^{-1} Q (1 + R_{t+t_0} S)^{-1} f(x)$. Hence we can rewrite (3.9) in the following way: If $0 \leq t < c_1 + c_2$, then

$$\lim_{n \to \infty} n^{\frac{3}{2}} A_n^{(t)} f(x) = (1 + SR_t)^{-1} Q (1 + R_t S)^{-1} f(x). \tag{3.10}$$

Comparing $\overline{S}$, $\overline{R}_t$ with S, R_t, we easily see $\|\overline{SR}_t\| \leq \|SR_t\|$. Thus $0 < c_1 \leq c_2$. From this fact, the method on the above can be iterated infinitely often. Therefore (3.10) holds for all $t \geq 0$. This finishes the proof of Main Lemma. $\square$

4. Proof of Theorem 1

In this section functions f, h and W are assumed to satisfy the conditions (i), (ii) and (iii) in Theorem 1. We devide the proof into three parts. For any $K > 0$, set

$$\begin{aligned} L_n &= E\left[f(e^{S_n})h\left(\sum_{i=1}^{n} W(S_i)\right)\right] \\ &= E\left[f(e^{S_n})h\left(\sum_{i=1}^{n} W(S_i)\right); M_n \le K\right] + E\left[f(e^{S_n})h\left(\sum_{i=1}^{n} W(S_i)\right); M_n > K\right] \\ &= H_n(K) + J_n(K). \end{aligned}$$

STEP 1. For any $K > 0$, the finite limit $\lim_{n\to\infty} n^{3/2} H_n(K)$ exists.

Proof. Let $K > 0$ be fixed. From the conditions (c) and (iii), we can choose two numbers μ and δ such that $0 < \delta < \mu < \min(\alpha, \varepsilon)$ and $E(e^{\delta\xi}) < \infty$. We easily see $f(e^x) \in C(-\delta)$ and $W \in C(-\mu)$. From the conditions of h, there exists a sequence of functions $\{h_m\}_{m\ge1}$ satisfying $\sup_{x\ge0} |h(x) - h_m(x)| \le m^{-1}$ and $h_m(x)$ is a finite linear combinations of $\{e^{-nx}\}_{n\ge0}$. Hence by Main Lemma we can define C_m as follows.

$$C_m = \lim_{n\to\infty} n^{\frac{3}{2}} E\left[f(e^{S_n})h_m\left(\sum_{i=1}^{n} W(S_i)\right); M_n \le K\right].$$

It is obvious that $\{C_m\}_{m\ge1}$ is a Cauchy sequence. Set $C_\infty = \lim_{m\to\infty} C_m$. Then

$$\begin{aligned} |n^{\frac{3}{2}} H_n(K) - C_\infty| \le{}& \frac{1}{m} n^{\frac{3}{2}} E\left[|f(e^{S_n})|; M_n \le K\right] \\ &+ \left| n^{\frac{3}{2}} E\left[f(e^{S_n})h_m\left(\sum_{i=1}^{n} W(S_i)\right); M_n \le K\right] - C_\infty \right|. \end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty} |n^{\frac{3}{2}} H_n(K) - C_\infty| \le \frac{a}{m} + |C_m - C_\infty| \longrightarrow 0 \quad \text{as } m \to \infty,$$

where $a = \lim_{n\to\infty} n^{3/2} E[|f(e^{S_n})|; M_n \le K]$. Thus we obtain the assertion of Step 1. □

STEP 2. $\limsup_{K\to\infty} \limsup_{n\to\infty} n^{3/2} |J_n(K)| = 0$.

Proof. From the condition (iii), there exist positive constants A, B and K which satisfy the following: $|f(x)| \le Ax^\alpha$, $|h(x)| \le Ax^{-\beta}$ for $x \ge 0$, and

$W(x) \geq Be^{\eta x}$ for $x \geq K$. Hence we have $|f(e^{S_n})| \leq Ae^{\alpha S_n}$ and the inequality, $|h(\sum_{i=1}^{n} W(S_i))| \leq AB^{-\beta}e^{-\beta\eta M_n}$, holds on $(M_n > K)$. Applying these estimates to $J_n(K)$ and using (2.1), we have

$$\begin{aligned}|J_n(K)| &\leq \mathrm{const}E(e^{\alpha S_n - \beta\eta M_n}; M_n > K)\\ &\leq \mathrm{const}\sum_{j=0}^{n}\psi_{n-j}(\alpha)\int_K^\infty e^{-bx}du_j(x),\end{aligned}$$

where const $= A^2B^{-\beta}$, $b = \beta\eta - \alpha > 0$. In view of (1) (2) of Lemma 4, we apply (2) of Lemma 3 to the last term of the above.

$$\limsup_{n\to\infty} n^{\frac{3}{2}}|J_n(K)| \leq \mathrm{const}\left\{\sum_{n=0}^{\infty}\psi_n(\alpha)\int_K^\infty e^{-bx}dU(x) + \psi(\alpha)\int_K^\infty e^{-bx}du(x)\right\}.$$

Since the last term goes to 0 as $K \to \infty$, Step 2 is proved. □

STEP 3. The finite limit $\lim_{n\to\infty} n^{3/2}L_n$ exists.

Proof. Without loss of generality, we assume that f and h are non-negative. By Step 1 we can define $H(K) = \lim_{n\to\infty} n^{3/2}H_n(K)$. Then $H(K)$ is a non-decreasing and bounded. Set $H = \lim_{K\to\infty} H(K)$. Since

$$n^{\frac{3}{2}}H_n(K) \leq n^{\frac{3}{2}}L_n \leq n^{\frac{3}{2}}H_n(K) + \sup_{m\geq n}\left(m^{\frac{3}{2}}J_m(K)\right),$$

taking account of Step 2 we easily see that $n^{3/2}L_n$ goes to H as $n \to \infty$. Finally Theorem 1 is established under the conditions (a), (b) and (c). □

REMARK. To prove Theorem 1 under the conditions (a), (c) and (d), we enumerate the modified points. Under the conditions (a) and (d), the local limit theorem

$$\lim_{n\to\infty}\left[\sup_{k\in\mathbb{Z}}\left|\sqrt{n}P(S_n = dk) - \frac{d}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(dk)^2}{2n\sigma^2}\right\}\right|\right] = 0$$

holds (See e.g. [3]), which implies, for $\theta > 0$

$$E(e^{-\theta S_n}; S_n > 0) \sim \frac{d}{\sqrt{2\pi}\sigma}\frac{e^{-d\theta}}{1 - e^{-d\theta}}n^{-\frac{1}{2}} \quad \text{as } n \to \infty.$$

This corresponds to Lemma 2. If U and V are defined by

$$\begin{aligned}U(x) &= \frac{d}{\sqrt{2\pi}\sigma}\sum_{0\leq j\leq i-1}u(dj) \quad (id \leq x < (i+1)d), \quad = 0 \quad (0 \leq x < d),\\ V(x) &= \frac{d}{\sqrt{2\pi}\sigma}\sum_{0\leq j\leq i}v(dj) \quad (id \leq x < (i+1)d),\end{aligned}$$

then (1) of Lemma 4 holds, and (2) (3) of Lemma 4 hold at every continuous point. Fix $K \in \mathbb{N}$. For a function f on $D \equiv \{id : i \leq K\}$ and $a \in \mathbb{R}$, set $\|f\|_a = \sup_{x \in D} e^{ax}|f(x)|$ and $C(a) = \{f : \|f\|_a < \infty\}$. Then x in the expectation symbol E_x takes value on D. Having these modifications, the proof of Theorem 1 under the conditions (a), (c) and (d) are carried out analogously. We give the following corollary used in the next section.

Corollary 10. *If f is positive on $(0, \infty)$ and h is positive on $[0, \infty)$, then the finite limit in Theorem 1 is positive.*

Proof. Set $\nu_n(x) = E[f(e^{S_n}); \sum_{i=1}^n W(S_i) \leq x]$. Then for any $t > 0$,

$$\int_0^\infty e^{-tx} d\nu_n(x) = E\left[f(e^{S_n}) \exp\left\{-t\sum_{i=1}^n W(S_i)\right\}\right].$$

Apply Theorem 1 and the extended continuity theorem for Laplace transform to the above. Then there exists a measure ν satisfying the following: For each $t > 0$,

$$\lim_{n\to\infty} n^{\frac{3}{2}} \int_0^\infty e^{-tx} d\nu_n(x) = \int_0^\infty e^{-tx} d\nu(x) < \infty.$$

On the other hand, (3.10) implies

$$\int_0^\infty e^{-tx} d\nu(x) \geq (1 + SR_t)^{-1} Q (1 + R_t S)^{-1} \hat{f}(0),$$

where $\hat{f}(x) = f(e^x)$. Letting $t \to 0$, we see $\nu(\infty) \geq Q\hat{f}(0) > 0$. The last inequality follows from Corollary 9. Hence we can choose a positive constant x such that $0 < \nu(x) < \infty$ and $\lim_{n\to\infty} n^{3/2}\nu_n(x) = \nu(x)$. By the assumptions of h, we have

$$E\left[f(e^{S_n}) h\left(\sum_{i=1}^n W(S_i)\right)\right] = \int_0^\infty h(s) d\nu_n(s) \geq \min_{0 \leq s \leq x} h(s)\nu_n(x),$$

which proves Corollary 10. □

5. Application to a random walk in a random medium

Let $\{p_i : i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with values in $[0, 1]$, and let $\mathcal{F}$ be the σ-field generated by the sequence $\{p_i\}$. A random walk in a random medium $\{p_i\}$ is a sequence of random variables $\{X_t : t = 0, 1, 2, \cdots\}$ which satisfy the following:

$$\begin{aligned} X_0 = 0, \quad & P(X_{t+1} = X_t + 1 | \mathcal{F}, X_0, X_1, \cdots, X_t = i) = p_i, \\ & P(X_{t+1} = X_t - 1 | \mathcal{F}, X_0, X_1, \cdots, X_t = i) = 1 - p_i. \end{aligned}$$

Set $\sigma_i = p_i/(1-p_i)$. In [7] it was shown that the condition $-\infty \leq E\log\sigma_0 < 0$ implies $\lim_{t\to\infty} X_t = -\infty$ a.s. Hence $\max_{t\geq 0} X_t$ is finite a.s. In [1] Afanas'ev showed the asymptotic behavior of the probability $P_n \equiv P(\max_{t\geq 0} X_t \geq n)$ under the condition $E\sigma_0 < 1$ which implies $-\infty \leq E\log\sigma_0 < 0$ by Jensen's inequality. We consider the same problem under the conditions

$$-\infty \leq E\log\sigma_0 < 0, \quad E\sigma_0 < \infty. \tag{5.1}$$

Theorem 2. *Let σ_0 satisfy* (5.1).

(1) *If $E\sigma_0\log\sigma_0 < 0$, then*

$$P_n \sim c_1(E\sigma_0)^n \qquad \text{as } n\to\infty.$$

(2) *If $E\sigma_0\log\sigma_0 = 0$ and $E\sigma_0\log^2\sigma_0 < \infty$, then*

$$P_n \sim c_2 n^{-1/2}(E\sigma_0)^n \quad \text{as } n\to\infty.$$

(3) *If $E\sigma_0\log\sigma_0 > 0$ and the distribution of $\log\sigma_0$ is continuous or supported on a lattice, then*

$$P_n \sim c_3 n^{-3/2}\gamma^n \qquad \text{as } n\to\infty,$$

where $\gamma = \min_{0\leq t\leq 1} E\sigma_0^t$ and c_1, c_2, c_3 are positive numbers independent of n. In case of (1) *and* (2), *$E\sigma_0 < 1$ holds automatically.*

Proof. Set $\zeta_i = \log\sigma_{i-1}$, $\widehat{S_n} = \sum_{i=1}^n \zeta_i$, $n\geq 1$. Then we have the following identity (See [1]).

$$P_n = E\rho\left(\rho + \sum_{i=1}^n e^{-\widehat{S_i}}\right)^{-1} \quad where \quad \rho = 1 + \sum_{i=1}^\infty \sigma_{-1}\sigma_{-2}\cdots\sigma_{-i} < \infty \quad a.s.$$

Since $\{p_i\}$ are i.i.d., $\{\zeta_i\}_{i\geq 1}$ are i.i.d. and independent of ρ. Set $h(x) = E\rho(\rho+x)^{-1}$. Then we can rewrite P_n using $h(x)$.

$$P_n = Eh\left(\sum_{i=1}^n e^{-\widehat{S_i}}\right).$$

Set $\phi(t) = E\exp t\zeta_1$. The condition (5.1) ensures that $\phi(t)$ is well defined on $[0, 1]$, $\phi''(t) > 0$ on $(0, 1)$ and $\lim_{t\searrow 0}\phi'(t) = E\log\sigma_0 < 0$. From the assumptions of (1) and (2), $\lim_{t\nearrow 1}\phi'(t) = E\sigma_0\log\sigma_0 \leq 0$. Since $\phi'(t)$ is monotone increasing on $(0,1)$, these estimates imply $\phi(1) = E\sigma_0 < 1$. Thus (1) and (2) are reduced to Afanas'ev's theorem in [1].

In case of (3), we have $\lim_{t\searrow 0}\phi'(t)<0$ and $\lim_{t\nearrow 1}\phi'(t)>0$. Therefore there exists a unique point $\tau\in(0,1)$ where $\phi(t)$ attains its minimum $\gamma<1$, i.e., $\phi'(\tau)=0$ and $\phi(\tau)=\gamma<1$. Set a proper distribution function

$$G(x)=\frac{1}{\gamma}\int_{-\infty}^{x}e^{\tau y}dP(\zeta_1\le y).$$

Let $\{\xi_i\}_{i\ge 1}$ be i.i.d. and $-\xi_1$ has the distribution function $G(x)$. By the assumption of (3) and the definition of $G(x)$, we see that ξ_1 satisfies the conditions in Section 1. In fact we obtain

$$Ee^{\theta\xi_1}=Ee^{(\tau-\theta)\xi_1}=\frac{\phi(\tau-\theta)}{\gamma}\quad\text{and}\quad E\xi_1=-\frac{\phi'(\tau)}{\gamma}=0.$$

The former of the above shows that the Laplace transform of ξ_1 exists in some neighborhood of the origin and $E|\xi_1|^k<\infty$ for all $k\in\mathbb{N}$. Put $S_0=0$ and $S_n=\sum_{i=1}^n\xi_i$, $n\ge 1$. We express P_n by S_i.

$$P_n=Eh\left(\sum_{i=1}^n e^{-\widehat{S_i}}\right)=\gamma^n E\left[e^{\tau S_n}h\left(\sum_{i=1}^n e^{S_i}\right)\right]. \tag{5.2}$$

Let a number β satisfying $\tau<\beta<1$ and $\phi(\beta)<1$ be fixed. By the definitions of ρ and $\phi(t)$, $B\equiv E\rho^\beta\le\sum_{n=0}^\infty\phi(\beta)^n<\infty$. Using Chebyshev's inequality, $P(\rho>y)\le By^{-\beta}$ for $y>0$. Applying integration by parts to $h(x)$ and using the above result, we have

$$\begin{aligned}h(x)&=\int_0^\infty\frac{x}{(y+x)^2}P(\rho>y)dy\\&\le\int_0^\infty\frac{x}{(y+x)^2}By^{-\beta}dy\\&=Bx^{-\beta}\int_0^\infty\frac{t^{-\beta}}{(1+t)^2}dt.\end{aligned}$$

Hence for $x>0$,

$$0<h(x)\le\frac{\pi\beta}{\sin\pi\beta}Bx^{-\beta}. \tag{5.3}$$

Taking account of (5.2) and (5.3), we can apply Theorem 1 and Corollary 10 to $P_n\gamma^{-n}$. Consequently we have the following positive finite limit

$$c_3\equiv\lim_{n\to\infty}n^{3/2}\gamma^{-n}P_n=\lim_{n\to\infty}n^{3/2}E\left[e^{\tau S_n}h\left(\sum_{i=1}^n e^{S_i}\right)\right].$$

This finishes the proof of (3). □

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